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ROCKET BOOSTER CONTROL

SECTION 9

TIME OPTIMAL CONTROL WITH AMPLITUDE AND

RATE LIMITED CONTROLS

NASA Contract NASw-563

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MILITARY PRODUCTS GROUP   RESEARCH DEPARTMENT

ROCKET BOOSTER CONTROL

SECTION 9

TIME OPTIMAL CONTROL WITH AMPLITUDE AND
RATE LIMITED CONTROLS

NASA Contract NASw-563

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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AMPLITUDE AND RATE LIMITED
CONTROLS[*]

W. W. Schmaedeke[+] and D. L. Russell[‡]

ABSTRACT

Necessary conditions leading to a method for the determination of bounded control amplitude and amplitude rate time optimal control trajectories by backing out of the origin are developed. The backing out procedure requires choosing the response time, the unaugmented system adjoint vector at the response time, the rate limited control variable amplitudes at the response time, and the rate limited control variable amplitudes at the initial time. A set of consistency conditions on the control variables are then used to determine the allowable control variable trajectories from a finite set of possibilities. The state trajectories including the state at the initial condition can be determined in the usual manner from the control variable trajectories.

INTRODUCTION

It has long been recognized that the maximum principle of Pontryagin would have to be modified to allow for controls whose switching rates were finite, due either to inertial or other factors.

The first insight into the form of the resulting theory was provided by Birch and Jackson in their 1959 paper, reference

2, although they were discussing quite a different problem.

The first discussion of the problem together with a set of necessary conditions characterizing the optimal controllers was provided by Chang. Several of the results in this paper were indicated by him in reference 1. The proofs herein are rigorous however, whereas Chang's are heuristic. The aspect of the problem that is new in the treatment herein is the requirement that solutions of the augmented adjoint equations be differentiable on the whole interval (0,T) instead of merely piecewise differentiable on so called "pang" intervals. It is this requirement which allows the "pang" intervals to be located. To be more specific, it is shown that the optimal control is either at extreme amplitude or extreme velocity. The sub-intervals of (0,T) over which this behavior occurs can be determined if appropriate initial and final conditions are given.

PRELIMINARIES

Consider the linear differential equation

$$\dot{x} = A(t)x + B(t)u + c(t) \tag{1}$$

where A is an nxn matrix, B is an nxm matrix, and c is an n-vector. The elements of A, B, and c are bounded continuous functions of time on an interval I under consideration. It is supposed that there are no constraints on the phase variables x(t) other than the given initial point and the target, and that the controls u(t) have components that are bounded in amplitude and rate. The class of admissible controls is defined as all vector functions u(t) defined on various subintervals of I whose

components satisfy

$$a_{1i}(t) \leq u_i(t) \leq a_{2i}(t) \quad i = 1,\ldots,m$$

$$b_{1i}(t) \leq \dot{u}_i(t) \leq b_{2i}(t) \quad i = 1,\ldots,k;$$

(2)

where $k \leq m$. The functions $a_{1i}(t)$, $a_{2i}(t)$, $b_{1i}(t)$, and $b_{2i}(t)$ are bounded continuous functions with the further assumption that

$$b_{1i} < \dot{a}_{2i} < b_{2i}, \; b_{1i} < \dot{a}_{1i} < b_{2i}$$

(3)

at all times at which the a's are differentiable (which is assumed to be almost everywhere).

By defining new controls v_i for $i = 1,\ldots,m$ with $v_1 = \dot{u}_1$, $v_2 = \dot{u}_2,\ldots,v_k = \dot{u}_k$, $v_{k+1} = u_{k+1},\ldots,v_m = u_m$ and new phase variables z_i for $i = 1,\ldots,n + k$ with $z_1 = x_1,\ldots,z_n = z_n$, $z_{n+1} = u_1,\ldots,z_{n+k} = u_k$ the system

$$\dot{z} = Fz + Gv + h$$

(4)

is obtained where

$$F = \begin{bmatrix} a_{11} & \cdots & a_{1n} & \vdots & b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & \vdots & b_{n1} & \cdots & b_{nk} \\ \hline 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 \end{bmatrix} \equiv \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & \cdots & 0 & \vdots & b_{1,k+1} & \cdots & b_{1m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \vdots & b_{n,k+1} & \cdots & b_{nm} \\ \hline 1 & 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \vdots & & \\ \vdots & & \ddots & 0 & \vdots & \\ \vdots & & & 0 & \vdots & \\ 0 & & & 0 \; 1 & \vdots & 0 & \cdots & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & B_1 \\ I_k & 0 \end{bmatrix}$$

(5)

$$h = \begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and where A is the original system's $n \times n$ coefficient matrix, B_0 is an $n \times k$ matrix whose k columns are the first k columns of the original control coefficient matrix B, B_1 is an $n \times (m-k)$ matrix whose columns are the remaining $(m-k)$ columns of B, and I_k is a

kxk identity matrix. The zero matrices are blocks of zeros of the appropriate dimension to make F have dimension $(n+k) \times (n+k)$ and G have dimension $(n+k) \times m$; the number of zeros in h is k so that h is an $(n+k)$-vector.

The system (4) is now in a bounded phase setting, that is,

$$a_{1i} \leq z_{n+i}(t) \leq a_{2i}; \qquad i = 1,\ldots,k \qquad (6)$$

(this is the bounded phase constraint); furthermore, the bounds on the amplitude of the new control vector $v(t)$ are given by

$$b_{1i} \leq v_i(t) \leq b_{2i}; \quad i = 1,\ldots,k$$

$$a_{1j} \leq v_j(t) \leq a_{2j}; \quad j = k+1,\ldots,m. \qquad (7)$$

NECESSARY CONDITIONS FOR OPTIMAL CONTROLS

For each time τ with $[0,\tau]$ contained in I, the set of all admissible controls on $[0,\tau]$ together with their corresponding responses is considered. The set of attainability $K(\tau)$ is the set of all points $x(\tau)$ in R^n which are terminal points of these response trajectories, i.e., if $x(t)$ is the response to the control $u(t)$ defined on the interval $[0,\tau]$, then the point $x(\tau)$ is to be included in the set $K(\tau)$. It can be shown that $K(\tau)$ is a compact convex subset of R^n. By considering the collection Σ of all non-empty compact subsets of R^n with the distance $d(C_1,C_2)$ between two such subsets C_1 and C_2 defined to be the infimum of all numbers d such that C_1 lies in the d-neighborhood of C_2 and C_2 lies in the d-neighborhood of C_1, Σ becomes a complete metric space. Now the set $K(\tau)$ is a compact subset of R^n and belongs

to Σ; furthermore, $K(\tau)$ is continuous in τ, (i.e. $K(\tau_1)$ and $K(\tau_2)$ are close in the above metric sense if τ_1 and τ_2 are close together). If $x(0)$ is not in the target, then as τ increases, there is a first time T at which the set $K(\tau)$ comes in contact with the target. Thus the optimal response $\hat{x}(\tau)$ has its terminal point $\hat{x}(T)$ on the boundary of $K(T)$.

Properties of controls $u(t)$ on an interval $[0,t_1]$ whose responses hit the boundary of $K(t_1)$ will be examined. To this end the following definitions are made:

DEFINITION 1. The linear control process (4) subject to (6) and (7) is considered. An admissible control $v(t)$ on the interval $[0,T]$ is called an extremal control $\hat{v}(t)$ in case there exists a non-trivial solution $\hat{\psi}(t)$ of the adjoint equations*

$$\dot{\psi} = - F'\psi$$

such that

$$\int_0^T \psi'(s) \begin{bmatrix} 0 & B_1(s) \\ I & 0 \end{bmatrix} \hat{v}(s)ds = \max_{v(s)} \int_0^T \psi'(s) \begin{bmatrix} 0 & B_1(s) \\ I & 0 \end{bmatrix} v(s)ds$$

where the maximum is taken over all admissible controllers $v(s)$.

LEMMA 1. A control $\hat{v}(t)$ on $[0,T]$ is extremal if and only if the corresponding response $z(t)$ has its terminal point $\hat{z}(T)$ on the boundary of $K(T)$.

PROOF: Assume $\hat{v}(t)$ is such that $\hat{z}(T)$ lies in the boundary of $K(T)$.

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* A prime on a vector or matrix means the transpose of that vector or matrix.

Then let π be a support plane to $K(T)$ at the point $\hat{z}(T)$, and let $\boldsymbol{\eta}$ be an outward normal to $K(T)$ at the point $\hat{z}(T)$. Then

$$\boldsymbol{\eta}' [\hat{z}(T) - z(T)] \geq 0 \qquad (8)$$

for any point $z(T)$ belonging to $K(T)$. Now let $\Lambda(t,s)$ be a fundamental solution of the homogeneous equation corresponding to (4) with $\Lambda(s,s) = I$, the $(n+k) \times (n+k)$ identity, and consider the variation of parameters formula for a solution of (4):

$$\hat{z}(T) = \Lambda(T,0)z_0 + \int_0^T \Lambda(T,0)\Lambda^{-1}(s,0) \begin{bmatrix} 0 & B_1(s) \\ I & 0 \end{bmatrix} \hat{v}(s)ds$$

$$+ \int_0^T \Lambda(T,0)\Lambda^{-1}(s,0)h(s)ds. \qquad (9)$$

Hence

$$\boldsymbol{\eta}' [\hat{z}(T) - z(T)] = \boldsymbol{\eta}' \int_0^T \Lambda(T,s) \begin{bmatrix} 0 & B_1(s) \\ I & 0 \end{bmatrix} [\hat{v}(s)-v(s)]ds \geq 0. \qquad (10)$$

Let $\psi(s)$ be a particular solution of the adjoint equations by defining

$$\psi'(s) = \boldsymbol{\eta}' (T,s).$$

Then

$$\int_0^T \psi'(s) \begin{bmatrix} 0 & B_1(s) \\ I & 0 \end{bmatrix} [\hat{v}(s)-v(s)]ds \geq 0, \qquad (11)$$

i.e., $\hat{v}$ is an extremal control.

The other case namely, if $\hat{v}(t)$ is extremal, is proven by beginning with equation (11) and proceeding backwards through the proof of the first case. This completes the proof.

According to Lemma 1, these extremal controls are candidates for the optimal controls since previous remarks have established that an optimal control has a response whose terminal point lies on the boundary of $K(T)$.

It will be convenient to decompose the adjoint vector $\psi'(s)$ as follows

$$\psi'(s) = (\theta'(s), \phi'(s)) \tag{12}$$

where $\theta(s)$ is an n-vector and $\phi(s)$ is a k-vector. Then (11) becomes

$$\int_0^T [\phi'(s), \theta'(s) B_1(s)][\hat{v}(s) - v(s)] ds \geq 0. \tag{13}$$

$v(s)$ is decomposed by defining

$$v(s) = \begin{bmatrix} \tilde{v}(s) \\ \underaccent{\tilde}{u}(s) \end{bmatrix} \tag{14}$$

where $\tilde{v}(s)$ is a k-vector whose components are $\tilde{v}_1(s) \equiv v_1(s), \ldots, \tilde{v}_k(s) \equiv v_k(s)$, and where $\underaccent{\tilde}{u}(s)$ is an (m-k)-vector whose components are $\underaccent{\tilde}{u}_1(s) \equiv u_{k+1}(s), \ldots, \underaccent{\tilde}{u}_{(m-k)}(s) \equiv u_m(s)$. Also, for later use, $\tilde{u}(s)$ is defined as a k-vector whose components are $\tilde{u}_1(s) \equiv u_1(s), \ldots, \tilde{u}_k(s) = u_k(s)$. Then (13) may be written as

$$\int_0^T \phi'(s)[\hat{\tilde{v}}(s) - \tilde{v}(s)] ds + \int_0^T \theta'(s) B_1(s) [\hat{\underaccent{\tilde}{u}}(s) - \underaccent{\tilde}{u}(s)] ds \geq 0. \tag{15}$$

It is now possible to refine Lemma 1 as follows:

LEMMA 2. An extremal control $\hat{v}(t)$ must be such that its first k components (represented by $\hat{\tilde{v}}(t)$) satisfy

$$\int_0^T \phi_i'(s) [\hat{\tilde{v}}_i(s) - \tilde{v}_i(s)] ds \geq 0 \tag{16}$$

for $i = 1, \ldots, k$, and for all $\tilde{v}_i(s)$ which are admissible i^{th} components of admissible controls $v(s)$; furthermore, the remaining

m-k components of $\hat{v}(t)$ (represented by $\hat{\underset{\sim}{u}}(t)$ satisfy

$$\int_0^T [\theta'(s)B_1(s)]_i [\hat{\underset{\sim}{u}}_i(s) - \underset{\sim}{u}_i(s)]ds \geq 0, \tag{17a}$$

or equivalently

$$\int_0^T [\theta'(s)B_1(s)]_i [\hat{u}_{k+i}(s) - u_{k+i}(s)]ds \geq 0 \tag{17b}$$

for $i = 1, \ldots, m-k$, and for all admissible control components $u_{k+i}(s)$.

PROOF: Let a particular choice of $v(s)$ be made as follows: $v_j(s) \equiv \hat{v}_j(s)$ for $j \neq i$ and let $v_i(s)$ be merely admissible. Then $\hat{v}(s) - v(s)$ has at most one non-zero component namely, $\hat{v}_i(s) - v_i(s)$. With this choice for $v(s)$, the second integral in (15) vanishes and condition (16) of the lemma is established. Condition (17a) and its equivalent condition (17b) are proved in a similar manner.

Returning to equation (8) η is decomposed as follows:

$$\eta' = (\lambda', \zeta') \tag{18}$$

where λ is an n-vector and ζ is a k-vector. According to the definition of $\tilde{u}(t)$ in the remarks following equation (14), $z(T)$ may be written as

$$z(T) = \begin{bmatrix} x(T) \\ \tilde{u}(T) \end{bmatrix}$$

and (8) becomes

$$\lambda' [\hat{x}(T) - x(T)] + \zeta' [\hat{\tilde{u}}(T) - \tilde{u}(T)] \geq 0. \tag{19}$$

By utilizing the variation of parameters representation of a solution of (1) (with $E(t,s)$ as a fundamental solution matrix of the homogeneous equation where $E(s, s)$ is the nxn identity.

$$x(t) = E(t,0)x_0 + \int_0^t E(t,s)\, B(s)\, u(s)ds +$$

$$+ \int_0^t E(t,s)\, c(s)ds. \tag{20}$$

Now, noting that

$$\theta'(t) = \lambda'(t)E(t,s) \tag{21}$$

and substituting this and (20) into (19) there results

$$\int_0^T \theta'(s)\, B(s)[\hat{u}(s)-u(s)]ds + \zeta'[\hat{\tilde{u}}(T) - \tilde{u}(T)] \geq 0, \tag{22}$$

where u(t) is any admissible control vector

Lemma 3 is established in a manner identical to that used for Lemma 2.

LEMMA 3. An extremal control v(s) must be such that its first k components (represented by $\dot{\hat{u}}_1(t),\ldots,\dot{\hat{u}}_k(t)$), when integrated, yield control components $\hat{u}_1(t),\ldots,\hat{u}_k(t)$ which satisfy

$$\int_0^T [\theta'(s)\, B(s)]_i [\hat{\dot{u}}_i(s)-u_i(s)]ds + \zeta_i[\hat{u}_i(T)-u_i(T)] \geq 0 \tag{23}$$

for i = 1,...,k and all admissible components $u_i(t)$; furthermore, they must satisfy

$$\int_0^T [\theta'(s)\, B(s)]_i [\hat{u}_i(s)-u_i(s)]ds \geq 0 \tag{24}$$

for i = k+1,...,m and all admissible components $u_i(t)$.

REMARK 1. It is observed that the entire matrix B appears in the integrand whereas in Lemma 2, the matrix was B_1, i.e., the last (m-k) columns of B. Thus (24) is equivalent to (17) because

$$[\theta'(s)\, B(s)]_{k+j} = [\theta'(s)\, B_1(s)]_j \tag{25}$$

for $j = 1,\ldots,m-k$.

Some qualitative properties of extremal controls will now be established. These are also necessary conditions for an optimal control. These conditions will be more conveniently phrased in terms of $u(t)$ rather than $v(t)$.

THEOREM 1. Let $\hat{u}(t)$ be an extremal control for the system (1). If $\hat{u}_1(t)$ is at its upper limit during an interval of time, then the function $[\theta'(s) B(s)]_1 \geq 0$ on that interval. Also, if $\hat{u}_1(t)$ is at its lower limit during an interval of time, then $[\theta'(s) B(s)]_1 \leq 0$ on that interval.

PROOF: Let $\hat{u}_1(t) = a_{21}(t)$ on an interval $[t_1,t_2]$ and suppose that $[\theta'(\tau) B(\tau)]_1 < 0$ at some point τ in $[t_1, t_2]$. By continuity, there is an interval $[\tau_1, \tau_2]$, containing τ in its interior, on which $[\theta'(t) B(t)]_1 < 0$. Consider equation (23) with $u_1(t)$ chosen so that

$$\hat{u}_1(t) - u_1(t) = \begin{cases} 0 \text{ outside of } [\tau_1, \tau_2] \\ \\ \mu(t) > 0 \; [\tau_1, \tau_2] \end{cases} \tag{26}$$

Then from (23) $\left\{\text{noting } u_1(T) - \hat{u}_1(T) = 0\right\}$

$$\int_{\tau_1}^{\tau_2} [\theta'(s) B(s)]_1 \, \mu(s) \, ds \geq 0. \tag{27}$$

But the integrand is negative on the entire interval and this is a contradiction. The remainder of the theorem is proved in a similar manner. Q.E.D.

Now consider again the adjoint equations for (4):

$$\dot{\psi} = - F' \psi \tag{28}$$

or, in terms of θ and ϕ

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} = - \begin{bmatrix} A' & 0 \\ B'_o & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix} \tag{29}$$

By performing the indicated matrix multiplication the two sets of equations are obtained:

$$\dot{\theta} = -A' \ \theta \tag{30}$$

$$\dot{\phi} = - B'_o \theta \tag{31}$$

Notice that θ corresponds to the adjoint vector of the original system (1) whereas ϕ, corresponding to the augmented coordinates of the adjoint vector, is a trivial linear system in that no components of ϕ appear on the right sides.

Given a fundamental solution $E(t, t_o)$ to equations (30), (with $E(t_o, t_o)$ = nxn identity) represent $\theta(t)$ may be represented by

$$\theta(t) = E(t, t_o) \ \theta_o \tag{32}$$

Then (31) yields

$$\phi(t) - \phi(t_o) = \int_{t_o}^{t} -B'_o(s) \ E(s, t_o) \ \theta_o ds \tag{33}$$

In the following, a technique for utilizing $\phi(t)$ in the construction of optimal trajectories will be developed.

DEFINITION 1. Let $u_1(t)$ be an admissible component of the control vector for (1) or (4), and define an interval of type B as a maximal closed subinterval of the interval $[0,T]$ whereon $u_1(t)$ is extremal, i.e., assumes maximum or minimum amplitude throughout the whole subinterval.

DEFINITION 2. An interval of type P_1 for $u_i(t)$ is defined to be a maximal closed interval in the interior of which $u_i(t)$ is not extreme valued, i.e., u_i assumes neither its maximum nor its minimum amplitude at any point in the interior of the interval. Note that if $P_1 \neq [0,T]$, then $\hat{u}_i$ is extreme at one end (or both) of P_1.

DEFINITION 3. An interval of type P_2 for $u_i(t)$ is defined to be a maximal subinterval of $[0,T]$ whereon $u_i(t)$ is not extreme and whereon $\dot{u}_i(t)$ is at one of its extremes, but not both.

REMARK 2. The interval $[0,T]$ can be decomposed into non-overlapping intervals of type B or type P_1 whose union is $[0,T]$.

THEOREM 2. Let the system (1) be normal and consider an extremal control vector $\hat{u}(t)$ for (1). For each $i = 1,\ldots,m$ it is the case that on an interval of type P_1 for $\hat{u}_i(t)$, either $\dot{\hat{u}}_i(t)$ is at its maximum value or its minimum value at every t at which $\dot{\hat{u}}_i(t)$ is defined.

PROOF : Let t belong to the interior of P_1 and assume that $\dot{\hat{u}}_i(t)$ is defined and is not extreme. Then, since $[\theta'(s) B(s)]_i$ is not zero on an interval by normality, we may assume further that t is such a point where $[\theta'(s) B(s)]_i$ is not zero. (This would eliminate a set of t in the interior of P_1 whose measure is zero). By continuity, $[\theta'(s) B(s)]_i$ is of one sign on an interval about the point t under consideration. For definiteness, assume $[\theta'(t) B(t)]_i < 0$. Then since $\hat{u}_i(t)$ is not extreme and since $\dot{\hat{u}}_i(t)$ is not extreme, an admissible control $u_i(s)$ is constructed as follows: Let M_1 be a line through the point $(t, \hat{u}_i(t))$ whose

slope is $b_{11}(t)$. (For simplicity, it is assumed that $b_{11}(s) \leq 0 \leq b_{21}(s)$ where both equalities do not hold simultaneously. Other cases would be treated similarly.) For a given $\delta > 0$, let $m_1(\delta)$ be a line through $(t, \hat{u}_i(t))$ whose slope is equal to the minimum of $b_{21}(s)$ on the interval $[t, t + \delta]$ and let $m_2(\delta)$ be a line through the same point whose slope is the maximum of $b_{11}(s)$ on the interval $[t, t + \delta]$. Now let $\delta_{01} > 0$ be chosen so small that the line $m_1(\delta_{01})$ lies entirely between* the curve $\hat{u}_i(s)$ and the line M_1 in the interval $[t, t + \delta_{01}]$ and let $\delta_{02} > 0$ be chosen so small that the line $m_2(\delta_{02})$ lies between the curve $\hat{u}_i(s)$ and the line M_2 in the interval $[t, t + \delta_{02}]$. Let δ_0 denote the smaller of δ_{01} and δ_{02} and further be small enough that $[\theta'(s) B(s)]_i < 0$ in $[t, t + \delta_0]$ and let m_1 and m_2 be the lines corresponding to δ_0; see Fig. 1.

According to the previous construction, there is a segment S of the ordinate at $t + \delta_0$ which is cut out by the line m_1 and the curve $\hat{u}_i(s)$. Since $\hat{u}_i(t)$ is not equal to its minimum value $a_{11}(t)$, then by continuity, there is a point P on the segment S such that the line L through P parallel to m_1 will intersect the curve $\hat{u}_i(s)$ at the point R at a time τ in $(t, t + \delta_0)$ and such that it will intersect the line m_2 at a point Q at some time σ for which this intersection is above the height $a_{11}(\sigma)$. Now define $u_i(s)$ to be equal to $\hat{u}_i(s)$ for $s \leq t$, and $s \geq \tau$. On the interval $[t, \tau]$

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* m_1 may coincide with M_1; similarly m_2 may coincide with M_2.

define $u_1(s)$ to be the segment of m_2 between the point $(t, \hat{u}_1(t))$
and Q; and to be the segment of L between Q and R. Thus $u_1(s)$
is an admissible control satisfying the amplitude and the rate
bounds either by construction or because it is equal to $\hat{u}_1(s)$
which is assumed admissible.

Now equation (23) with the particular $u_1(s)$ just constructed
is considered. It is seen that

$$\int_t^\tau [\theta'(s) \, B(s)]_1 \, [\hat{u}_1(s) - u_1(s)]ds \geq 0 \qquad (34)$$

But $[\theta'(s) \, B(s)]_1 < 0$ in $[t, \tau]$ while $\hat{u}_1(s) - u_1(s) > 0$ in
(t, τ). This is a contradiction and hence the velocity of $\hat{u}_1(t)$
must be extreme. The proof goes through in the same way if it is
assumed that $[\theta'(t) \, B(t)]_1 > 0$. Q.E.D.

REMARK 3. It follows from theorem 2 that the interval $[0, T]$
is decomposable into subintervals of type B or type P_1 which
are nonoverlapping and whose union is $[0, T]$. In other words, the
optimal control is either at extreme amplitude or extreme velocity,
whenever the velocity is defined.

THEOREM 3. Let the system (1) be normal and consider an extremal
control vector $\hat{u}(t)$ for (1). For each $i = 1, \ldots, m$ it is the
case that if there is an interval of type P_1 for $\hat{u}_1$ such that
at least one of its endpoints say t*, is in the interior of
$[0, T]$, then for all t in the P_1 interval for which $u_1(t)$ is
defined: (i) $\phi_1(t) > \phi_1(t*)$ implies that $\hat{u}_1(t)$ is at its
maximum value; (ii) $\phi_1(t) < \phi_1(t*)$ implies that $\hat{u}_1(t)$ is at its

minimum value.

PROOF : From Theorem 2 $\dot{\hat{u}}_i(t)$ is at one extreme or the other in P_1 intervals. By hypothesis, since one of the endpoints of the P_1 interval under consideration is a point t* interior to (0, T) it may be assumed without loss of generality (w.l.o.g.) that the point t* is the right end of P_1. Furthermore, it is assumed w.l.o.g. that $\hat{u}_i(t*)$ is a minimum. Now let t be a point in P_1 at which $\dot{\hat{u}}_i(t)$ is defined and suppose that $\phi_i(t)$ is greater than $\phi_i(t*)$ but $\dot{\hat{u}}_i(t)$ is minimum (i.e., $b_{1i}(t)$). Figure 2 supplies the details.

$u_i(s)$ is chosen so that on [t, t + δ] it has maximum slope and lies above $\hat{u}_i(s)$ while it is parallel to $\hat{u}_i(s)$ from t + δ to some point τ > t*, (choose δ so small that τ < T). Then let $\hat{u}_i(s) = u_i(s)$ from τ to T.

Now from the construction of $u_i(s)$ and from equations (23)

$$\int_t^{t+\delta} [\theta'(s)\,B(s)]_i [\hat{u}_i(s) - u_i(s)]ds + \int_{t+\delta}^{t*} [\theta'(s)B(s)]_i [\hat{u}_i(s) - u_i(s)]ds$$

$$+ \int_{t*}^{\tau} [\theta'(s)\,B(s)]_i [\hat{u}_i(s) - u_i(s)]ds \geq 0 \qquad (35)$$

On [t + δ, t*] the function $\hat{u}_i(s) - u_i(s)$ has the constant value, say -ε. Thus the middle integral is

$$-\epsilon \int_{t+\delta}^{t*} [\theta'(s)\,B(s)]_i \, ds. \qquad (36)$$

Note that as δ approaches zero, the integral in (36) (ignoring the multiplicative factor ϵ) approaches

$$\int_t^{t*} [\theta'(s)\ B(s)]_i ds = -\int_t^{t*} \dot{\phi}_i(s)ds = \phi_i(t) - \phi_i(t*)$$

$$\underset{=}{\Delta}\ r > 0 \tag{37}$$

Choose δ_o so small that for all $\delta < \delta_o$

$$\frac{r}{2} < \int_{t+\delta}^{t*} [\theta'(s)\ B(s)]_i\ ds < \frac{3r}{2} \tag{38}$$

Now the first integral in (35) can be made small on the order of δ^2 as follows: since $|\hat{u}_i(s) - u_i(s)| \leq K_1\ |s - t|$ on $[t, t + \delta]$,

$$\left|\int_t^{t+\delta} [\theta'(s)\ B(s)]_i\ [\hat{u}_i(s) - u_i(s)]ds\right| \leq K_1 \int_t^{t+} \left|\ [\theta'(s)B(s)]_i\ |s-t|ds\right. \tag{39}$$

Letting $K_2 = \max_{[t,t+\delta_o]} \left|[\theta'(s)\ B(s)]_i\right|$ yields

$$\left|\int_t^{t+\delta} [\theta'(s)\ B(s)]_i\ [\hat{u}_i(s) - u_i(s)]ds\right| \leq K_1\ K_2\ \frac{\delta^2}{2}\ . \tag{40}$$

An easier analysis applies to the last integral; namely

$$\int_{t*}^{\tau} [\theta'(s)\ B(s)]_i\ [\hat{u}_i(s) - u_i(s)]ds\right| \leq K_3\ \epsilon\ (\tau - t*). \tag{41}$$

As a result of (41), (40), (38) and (35)

$$[K_1 \ K_2 \ \frac{\delta^2}{2} - \frac{r}{2} \ \epsilon + K_3 \ \epsilon \ (\tau - t^*)] \geq 0. \qquad (42)$$

It is next shown that δ is bounded above by a constant times ϵ. It is observed that $\dot{u}_1$ is greater than $\dot{\hat{u}}_1$ on $[t, \ t + \delta]$, hence their difference is not zero on $[t, \ t + \delta]$. Let the minimum of this difference be denoted by $c_1 > 0$, then

$$\int_t^{t+\delta} [\dot{\hat{u}}_1(s) - \dot{u}_1(s)]ds \geq c_1 \qquad (43)$$

or

$$c_1 \ \delta \leq u_1(s) - \hat{u}_1(s) \int_t^{t+\delta}$$

$$= \hat{u}_1(t+\delta) - u_1(t+\delta) \qquad (44)$$

$$- [\hat{u}_1(t) - u_1(t)]$$

$$= \epsilon$$

Thus $c_1 \ \delta \leq \epsilon$ or $\delta \leq c_2 \ \epsilon$ where $c_2 > 0$. Applying this result to (42) yields

$$K_1 \ K_2 \ c_2^2 \ \epsilon^2 - \frac{r}{2} \ \epsilon + K_3 \ \epsilon \ (\tau - t^*) \geq 0 \qquad (45)$$

or

$$\epsilon \ (K_1 \ K_2 \ c_2^2 \ \epsilon + K_3(\tau - t^*) - \frac{r}{2}) \geq 0. \qquad (46)$$

But (46) is a contradiction because for δ sufficiently small, ϵ and $\tau - t^*$ can be made arbitrarily small which means the quantity in parantheses is negative. Thus it has been shown that when $\phi_1(t) > \phi_1(t^*)$ then $\dot{\hat{u}}_1(t)$ is maximum (where it is defined). A similar proof will show that $\phi_1(t) < \phi_1(t^*)$ implies $\dot{\hat{u}}_1(t)$ is minimum.

REMARK 4. Note that intervals of type P_2 coincide with intervals whereon the sign of $\phi_1(t) - \phi_1(t^*)$ is constant for appropriately chosen points t^*. It will be shown later that there are only a finite number of these points t^* for a given $\phi_1(t)$ and that it is possible in those cases where $u_1(t)$ is given at the final time as well as the initial time, to construct the family of extremal controls.

THEOREM 4 . Let the system (1) be normal and consider an extremal control vector $\hat{u}(t)$ for (1). For each $i = 1, \ldots, m$ it is the case that if the entire interval $[0,T]$ (where T is the minimal time of response) is of type P_1 for $\hat{u}_1(t)$, then there exists a constant c_1 such that if $\phi_1(t) - c_1 > 0$ then $\dot{\hat{u}}_1(t)$ is at its maximum value and if $\phi_1(t) - c_1 < 0$ then $\dot{\hat{u}}_1(t)$ is at its minimum value (assuming that $\dot{\hat{u}}_1(t)$ is defined at t). If there are at least two intervals of type P_2 contained in the interval of type P_1, then the value of the constant c_1 is equal to ϕ_1 evaluated at any of the interior endpoints of the type P_2 intervals.

PROOF:

CASE I: If the whole interval $[0,T]$ is of type P_2 the theorem is trivially true as the constant c_1 in this case may be chosen to

be the minimum or the maximum of the function $\phi_i(t)$ on $[0,T]$ depending on whether $\hat{u}_i$ is maximum or minimum.

CASE II: If there are at least two intervals of type P_2 contained in P_1 then let t* be an interior endpoint of a P_2 interval. Consider the case where $\hat{u}_i(t)$ is minimum to the left of t* and maximum to the right of t* (the other case with the maximum and minimum reversed would be treated similarly). Let t' be any interior point of $[0,T]$ which is not the endpoint of an interval of type P_2. Assume that $\phi(t') - \phi(t*) > 0$ but that $\hat{u}_i(t')$ is at its minimum. Let it be supposed that t' < t*. Then construct $u_i(t)$ on $[0,T]$ as follows: Let $u_i(t) = \hat{u}_i(t)$ for $t \leq t$; choose $\delta > 0$ so small that if $u_i(t)$ has maximum velocity on $[t', t + \delta]$, is parallel to $\hat{u}_i(t)$ on $[t' + \delta, t*]$, and has minimum slope for a suitable time duration to the right of t*, then the curve $u_i(t)$ will intersect the curve $\hat{u}_i(t)$ at some point τ to the right of t*. (This choice is possible because the slope of $\hat{u}_i(t)$ is maximum to the right of t*). Finally, let $u_i(t) = \hat{u}_i(t)$ on $[\tau,T]$. From here on, one proceeds exactly as in the proof of theorem 3 beginning with equation (35).

A METHOD FOR COMPUTATION OF EXTREMAL TRAJECTORIES FOR AMPLITUDE AND RATE LIMITED CONTROLS

The foregoing theorems will now be given a more useful interpretation. Since the case where a_1, a_2, b_{1i}, b_{2i}, $i = 1, \ldots k$ are constant

is of particular interest, as each condition on extremal trajectories is stated its specialization to this case will also be given. Each interval P_1 of type P_1 has a unique decomposition

$$P_1 = \overline{P}_{2_1} \cup \overline{P}_{2_2} \cup \ldots \cup \overline{P}_{2_r} \qquad (47)$$

into intervals P_{2_ρ} of type P_2 where $\overline{P}_{2_\rho} \cap \overline{P}_{2_{\rho+1}}$ consists of precisely one point, $\rho = 1, 2, \ldots, r-1$. The bar indicates topological closure. Let $\hat{u}_i$ be the ith component of the optimal control $\hat{u}$ corresponding to the function ϕ. In all that follows let P_1 be an interval of type P_1 for $\hat{u}_i$. Theorems 3 and 4 show that corresponding to the interval P_1 there is a constant c_i such that the subintervals P_{2_ρ} of P_1 coincide with those subintervals of P_1 whereon sign $(\phi_i(t) - c_i)$ is constant. Let

$$\text{sign } (P_{2_\rho}) \triangleq \text{sign } (\phi_i(t) - c_i), \quad t \in P_{2_\rho} \qquad (48)$$

Then (P_{2_ρ}) is set equal to the length of P_{2_ρ} and several cases are considered. Let τ_1, τ_2 be the endpoints of P_1.

CASE I: τ_1, τ_2 both belong to $(0,T)$. Then it is clear that $\hat{u}_i(\tau_1)$ and $\hat{u}_i(\tau_2)$ are both extremal. In fact

$$\hat{u}_i(\tau_1) = \begin{cases} a_{2i}(\tau_1) & \text{if } \text{sgn } (P_{2_1}) = -1 \\ a_{1i}(\tau_1) & \text{if } \text{sgn } (P_{2_1}) = +1 \end{cases} \qquad \text{(a)}$$

$$\hat{u}_i(\tau_2) = \begin{cases} a_{2i}(\tau_2) & \text{if } \text{sgn } (P_{2_r}) = +1 \\ a_{1i}(\tau_2) & \text{if } \text{sgn } (P_{2_r}) = -1 \end{cases} \qquad \text{(b)}$$

$$(49)$$

CASE II: Either τ_1 or τ_2, but not both belong to $(0,T)$. Then if $\tau_1 \in (0, T)$ 49(a) holds, if $\tau_2 \in (0,T)$ 49(b) holds. In each case the value of $\hat{u}_1$ at the other endpoint, i.e., either $\hat{u}_1(0)$ or $\hat{u}_1(T)$ must be specified in some other manner.

CASE III: $\tau_1 = 0$, $\tau_2 = T$. Then both $\hat{u}_1(0)$ and $\hat{u}_1(T)$ must be specified, neither 49(a) nor 49(b) hold.

It is possible to consider problems wherein neither $\hat{u}_1(0)$ nor $\hat{u}_1(T)$ are specified. In this case the procedure to be described below is not immediately applicable. This situation arises in the so-called interception problem. A remark on this will be made at the end of this paper.

The values which $\hat{u}_1(t)$ assumes at τ_1 and τ_2 lead to the following conditions merely by applying the fundamental theorem of calculus for absolutely continuous functions.

CONDITION 1

$$\sum_{\substack{P_{2_\rho} \\ \rho \ni \text{sgn}(P_{2_\rho}) = +1}} \int b_{21}(t)dt + \sum_{\substack{P_{2_\rho} \\ \rho \ni \text{sgn}(P_{2_\rho}) = -1}} \int b_{11}(t)dt = \hat{u}_1(\tau_2) - \hat{u}_1(\tau_1)$$

If the bounds on the control velocity are constant then

CONDITION 1a

$$\left[\sum_{\substack{\rho \\ \rho \text{ sgn}(P_{2_\rho}) = +1}} \ell(P_{2_\rho})\right] b_{21} + \left[\sum_{\substack{\rho \\ \rho \text{ sgn}(P_{2_\rho}) = -1}} \ell(P_{2_\rho})\right] b_{11} = \hat{u}_1(\tau_2) - \hat{u}_1(\tau_1)$$

The requirement that $u_i(t)$ shall not achieve an extreme value in the interior of P_1 leads to

CONDITION 2   For each σ such that $1 \leq \sigma < r$ (this set could be void)

$$\sum_{\substack{\rho \ni \text{sgn} (P_{2_\rho}) = +1 \\ \rho \leq \sigma}} \int P_{2_\rho} b_{2i}(t)dt \quad + \quad \sum_{\substack{\rho \ni \text{sgn} (P_{2_\rho}) = -1 \\ \rho \leq \sigma}} \int P_{2_\rho} b_{1i}(t)dt$$

$$\begin{cases} < a_{2i}(\tau_{2\sigma}) - \hat{u}_i(\tau_1) \\ > a_{1i}(\tau_{2\sigma}) - \hat{u}_i(\tau_1) \end{cases}$$

where $\tau_{2\sigma}$ is the right endpoint of the interval P_2. The inequalities (3) enable this testing procedure to be restricted to points $\tau_{2\sigma}$. Again if the bounds on the control velocity are constant

CONDITION 2a

$$\left[\sum_{\substack{\rho \ni \text{sgn} (P_{2_\rho}) = +1 \\ \rho \leq \sigma}} \ell (P_{2_\rho}) \right] b_{2i} \quad + \quad \left[\sum_{\substack{\rho \ni \text{sgn} (P_{2_\rho}) = -1 \\ \rho \leq \sigma}} \ell (P_{2_\rho}) \right] b_{1i}$$

$$\begin{cases} < a_{2i} - \hat{u}_i(\tau_1) \\ > a_{1i} - \hat{u}_i(\tau_1) \end{cases}$$

DEFINITION 4. Subintervals of $[0,T]$ on which any control $u_i(t)$ may be dfined so that Conditions 1 and 2 above are satisfied with $\hat{u}_i(t)$ replaced by $u_i(t)$, $\dot{u}_i(t) = b_{2i}(t)$ if sgn $P_{2_\rho} = +1$, $\dot{u}_i(t) = b_{1i}(t)$ if sgn $P_{2_\rho} = -1$, are called intervals of type P_3. Thus every interval of type P_1 is also of type P_3 by Theorems 3, 4. The converse need not hold since there may be no extension of $u_i(\tau)$ from the given interval of type P_3 into the entire interval $[0,T]$ as an extremal controller.

DEFINITION 5. A decomposition of $I = [0,T]$ into subintervals of types B and P_3 is called acceptable if the resulting control $u_i(t)$ is continuous and satisfies the preceding theorems on extremal controllers. The intervals of type P_3 then become intervals of type P_1 for $u_i(t)$.

The following theorem is of primary importance in establishing a procedure for computing extremal controllers.

THEOREM 5. Assume that $(\theta'B)_i(t)$ has at most finitely many zeroes on $[0,T]$ and the functions $a_{1i}(t)$ and $a_{2i}(t)$ are constants. Then there are at most finitely many possible intervals of type P_3, provided $u_i(0)$ and $u_i(T)$ are specified in advance.

PROOF: For a linear differential system the interval $[0,T]$ may be divided into finitely many subintervals in which $\phi_i(t)$ is monotone. To prove this, note that the negation implies that $(\theta'B)_i(t)$ has infinitely many zeroes in $[0,T]$, contrary to assumption. Thus the inverse function $t(\phi_i)$ of the function $\phi_i(t)$ consists of finitely many functions $t_1(\phi_i)$, ..., $t_s(\phi_i)$, each a

monotone function defined on some subinterval of

$\left[\min_{t\in[0,T]} \phi_i(t), \max_{t\in[0,T]} \phi_i(t)\right]$ and $\sigma_1 < \sigma_2$ implies that $t_{\sigma_1}(\phi_i) < t_{\sigma_2}(\phi_i)$

W.l.o.g. it may be assumed that $t_\sigma(\phi_1)$ is decreasing for odd σ

and increasing for even σ. (Figs. 3 and 4.) The other case is

handled similarly. Let $t_o(\phi_1) \equiv 0$, $t_{s+1}(\phi_1) \equiv T$.

The domain of definition of each $t_\sigma(\phi_1)$ is extended to all of

$\left[\min_{t\in[0,T]}\phi_1(t) \ \max_{t\in[0,T]}\phi_1(t)\right]$ by setting $t_\sigma(\phi_1)$ equal to $t_\sigma(\phi_1^*)$

where ϕ_1^* is the closest point to ϕ_1 where $t_\sigma(\phi_1^*)$ is already

defined (Fig. 5). For each of the finitely many pairs of indices

σ_1, σ_2, $0 \le \sigma_1 < \sigma_2 \le s + 1$, $g^i_{\sigma_1 \ \sigma_2}(\phi_1)$ is defined by

$$g^i_{\sigma_1 \ \sigma_2}(\phi_1) = \sum_{\sigma=\sigma_1+1}^{\sigma_2} \int_{t_{\sigma-1}(\phi_1)}^{t_\sigma} \beta_{\sigma i}(t)dt \qquad (50)$$

where

$$\beta_{\sigma i}(t) = \begin{cases} b_{2i}(t) & \text{if } \sigma \text{ is odd} \\ \\ b_{1i}(t) & \text{if } \sigma \text{ is even} \end{cases} \qquad (51)$$

Then it is easy to see that each $g^i_{\sigma_1 \ \sigma_2}(\phi_1)$ is a monotone

decreasing function of ϕ_1. (If $t_\sigma(\phi_1)$ were increasing for σ odd,

decreasing for σ even, then $g_{\sigma_1 \ \sigma_2}(\phi_1)$ would still be a monotone

decreasing function.)

It is clear by comparison of (50) with Condition 1 that an interval

of type P_3 can occur only when there exists a pair σ_1, σ_2 and a

value ϕ_1 such that

$$g^i_{\sigma_1 \; \sigma_2}(\phi_1) = u_i(t_{\sigma_2}(\phi_1)) - u_i(t_{\sigma_1}(\phi_1)) \tag{52}$$

where $u_i(t)$ is the control which must be defined on the interval according to the definition of an interval of type P_3. A number of cases are now considered:

If $\sigma_1 = 0$, $\sigma_2 = s + 1$, then we required in the hypotheses of this theorem that $u_i(0)$ and $u_i(T)$ be fixed. Thus $u_i(t_{\sigma_2}(\phi_1)) - u_i(t_{\sigma_1}(\phi_1))$ is a constant known beforehand.

If $\sigma_1 = 0$, σ_2 arbitrary > 0 then $u_i(t_{\sigma_1}(\phi_1))$ is fixed at a constant value known beforehand while $u_i(t_{\sigma_2}(\phi_1)) = a_{2i}(t_{\sigma_2}(\phi_1))$ or $a_{1i}(t_{\sigma_2}(\phi_1))$.

If $\sigma_2 = s + 1$ while σ_1 is arbitrary $< s + 1$ then $u_i(t_{\sigma_2}(\phi_1))$ is fixed at a constant value known beforehand while $u_i(t_{\sigma_1}(\phi_1)) = a_{2i}(t_{\sigma_1}(\phi_1))$ or $a_{1i}(t_{\sigma_2}(\phi_1))$.

If $0 < \sigma_1 < \sigma_2 < s + 1$ then $u_i(t_{\sigma_2}(\phi_1)) - u_i(t_{\sigma_1}(\phi_1))$ is one of the four functions $a_{\delta i}(t_{\sigma_2}(\phi_1)) - a_{\gamma i}(t_{\sigma_1}(\phi_1))$, $\delta = 1, 2$, $\gamma = 1, 2$.

Thus, since it was assumed the functions $a_{1i}(t)$, $a_{2i}(t)$ were constants, it has been shown that there are at most finitely many values which $u_i(t_{\sigma_2}(\phi_1)) - u_i(t_{\sigma_1}(\phi_1))$ may assume for each σ_1, σ_2. Since there are finitely many functions $g^i_{\sigma_1 \; \sigma_2}(\phi_1)$ and each of them is monotone, there are but finitely many instances wherein equation (52) may hold. This completes the proof of the theorem.

REMARK 5. In the case where $a_{2i}(t)$ and $a_{1i}(t)$ are not constant but vary with time, the finitely many values which we have shown

in the proof of the theorem may be equal to $u_i(t_{\sigma_2}(\phi_i)) - u_i(t_{\sigma_1}(\phi_i))$ must be replaced by the finitely many functions $u_i(t_{\sigma_2}(\phi_i)) - u(t_{\sigma_1}(\phi_i))$ themselves. Then the conclusion of the theorem remains valid if for each σ_1, σ_2 the function $g_{\sigma_1 \sigma_2}(\phi_i) - (u_i(t_{\sigma_2}(\phi_i)) - u_i(t_{\sigma_1}(\phi_i))$ has but finitely many zeroes on its domain of definition. It is difficult to give a reasonably general sufficient condition under which this holds so the restriction to the case where $a_{2i}(t)$ and $a_{1i}(t)$ are constant was made. Clearly the likelihood is very small that any of these functions would have infinitely many zeroes in any given application. Thus it is fairly safe to assume that there are but finitely many P_3 intervals even if $a_{1i}(t)$ and $a_{2i}(t)$ are time-varying but it should be kept in mind that this has not been established and it may be possible to construct pathological functions $a_{2i}(t)$, $a_{1i}(t)$ such that this would not be true.

REMARK 6. Note that the theorem also shows a method for finding the intervals of type P_3 since the functions $g_{\sigma_1 \sigma_2}(\phi_i)$ and the constants (or functions $u_i(t_{\sigma_2}(\phi_i)) - u_i(t_{\sigma_1}(\phi_i))$ are readily determined. An acceptable decomposition of $[0,T]$ into intervals of types B and P_3. Thus after having found all possible intervals of type P_3 (and the previous theorem assures us that in many cases this can be done), it remains only to find all acceptable decompositions of $[0,T]$, and hence all possible controls $u_i(t)$ which satisfy the first four theorems. There being only finitely

many of these the control $\hat{u}_1(t)$ which satisfies inequality (11)
can easily be found.: If no values are given beforehand for $u_1(0)$
and or $u_1(T)$ then these values could be varied and the above results
applied to each choice of those values to determine the best (in
the sense of (11)) set of values for $u_1(0)$ and or $u_1(T)$.

A short example illustrating the use of the above results is now
given.

AN EXAMPLE TO ILLUSTRATE THE
CONSTRUCTION OF AN EXTREMAL CONTROL

Let the time interval be $[0,2]$ and $(\theta' B)_1(t) = -\frac{5\pi}{2} \cos(\frac{5\pi}{2} t)$.
Then $\phi_1(t) = \sin(\frac{5\pi}{2} t)$. Suppose that $a_{21} = 1$, $a_{11} = -1$,
Require $\hat{u}_1(0) = 0$, $\hat{u}_1(2) = 0$. The extremal control $\hat{u}_1(t)$ on $[0,2]$
will be constructed. The method used will be graphic and will be
special to the constants a_{11}, a_{11}, b_{21}, b_{11} in this problem.
Its relationship to the immediately preceding discussion should
be clear, as well as generalizations to different constant bounds.
An interval of type P_3 occurs whenever it is possible to draw a
level line L through the graph of $\sin(\frac{5\pi}{2} t)$ so that the end-
points of L lie on the graph of $\sin(\frac{5\pi}{2} t)$ or else meet the lines
$t = 0$ or $t = 2$ and satisfies the following requirements: (Compare
with Conditions 1 and 2 above.)

1. If the endpoints of L are in $(0,2)$ then the sum S of
the lengths of those segments of L lying below $\sin(\frac{5\pi}{2} t)$ minus
the sum of the lengths of those segments of L lying above $\sin$
$(\frac{5\pi}{2} t)$ must be 2, -2, or 0. If L' is any segment of L such that
the left endpoints of L and L' coincide then, (a) If the first
segment of L' lies below $\sin(\frac{5\pi}{2} t)$ the sum S' of the lengths

of those segments of L' lying below sin ($\frac{5\pi}{2}$ t) minus the sum of the lengths of those segments of L' lying above sin ($\frac{5\pi}{2}$ t) must be <2 and >0. (b) If the first segment of L' lies above sin ($\frac{5\pi}{2}$ t) then the corresponding quantity must be >-2 and <0.

2. If t = 0 is an endpoint of L and the right hand endpoint of L belongs to (0, 2) then S = 1 or -1 and -1 < S' < 1 for any L'. A similar situation occurs if the left hand endpoint of L lies in (0, 2) and the right hand endpoint is at t = 2, but here L, L' are taken to have a common right hand endpoint.

3. If L stretches from t = 0 to t = 2 then S = 0 and -1 < S' < 1 for any L' having an endpoint in common with L.

The graph in Fig. 6 shows all possible intervals of type P_3 indicated by level lines through the graph. The only acceptable sequence of intervals consists of the single interval P_1 of type P_1 which is indicated in the figure. This is clear by inspection, using the results of the first four theorems. Fig. 7 shows the resulting extremal control $u_1(t)$.

CONCLUSIONS

Necessary conditions leading to a method for the determination of bounded control amplitude and bounded amplitude rate time optimal control trajectories by backing out of the origin were developed. It can thus be said the theory of bounded rate optimal control has been brought to the same stage of development as the theory of optimal control without rate or phase bounds.

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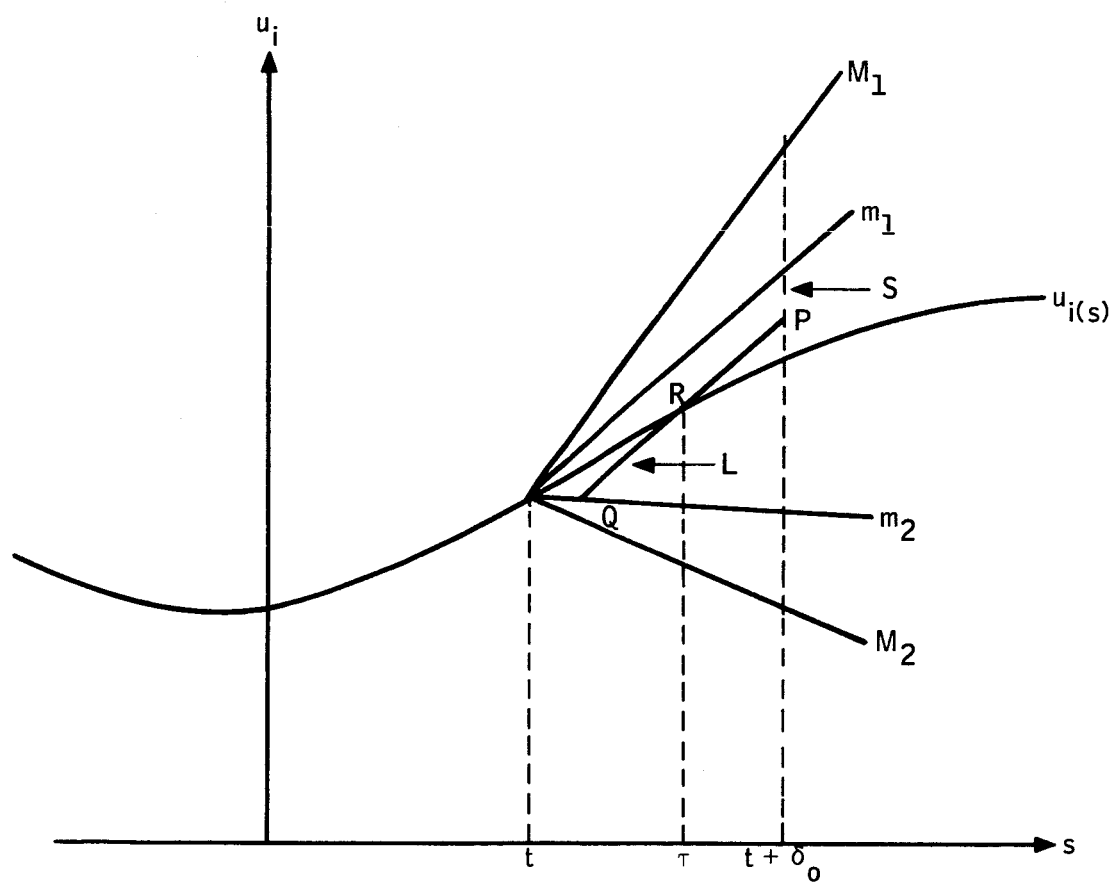
Figure 1. Construction of Admissible Varied Control to Prove $\overset{\star}{u}_i$ Extremal
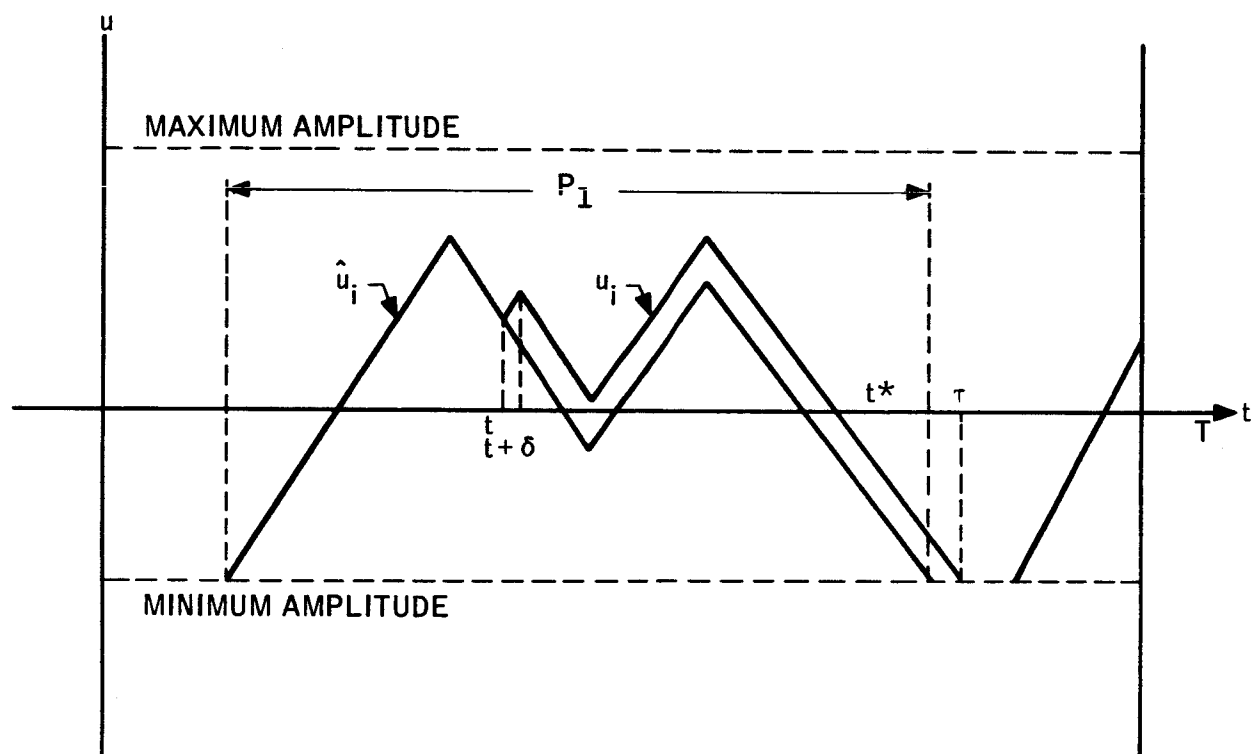
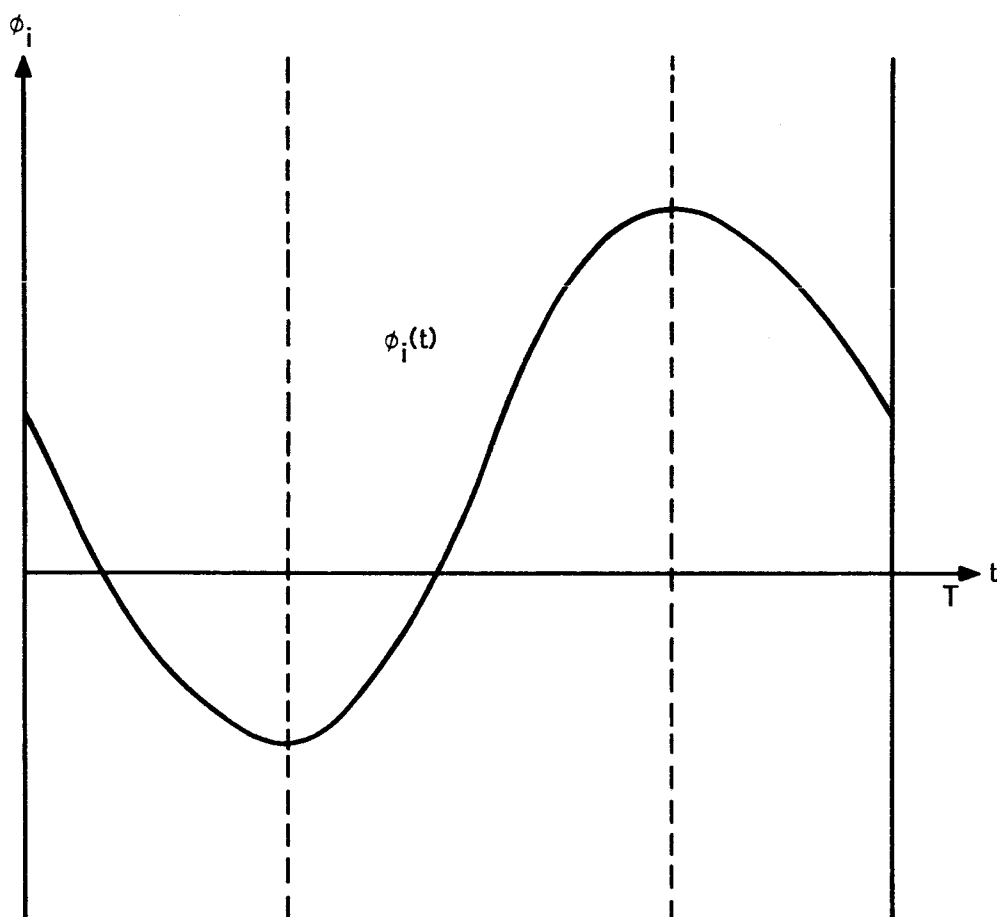Figure 2. Construction of Admissible Control in Proof of Necessary Conditions for P_1 Intervals

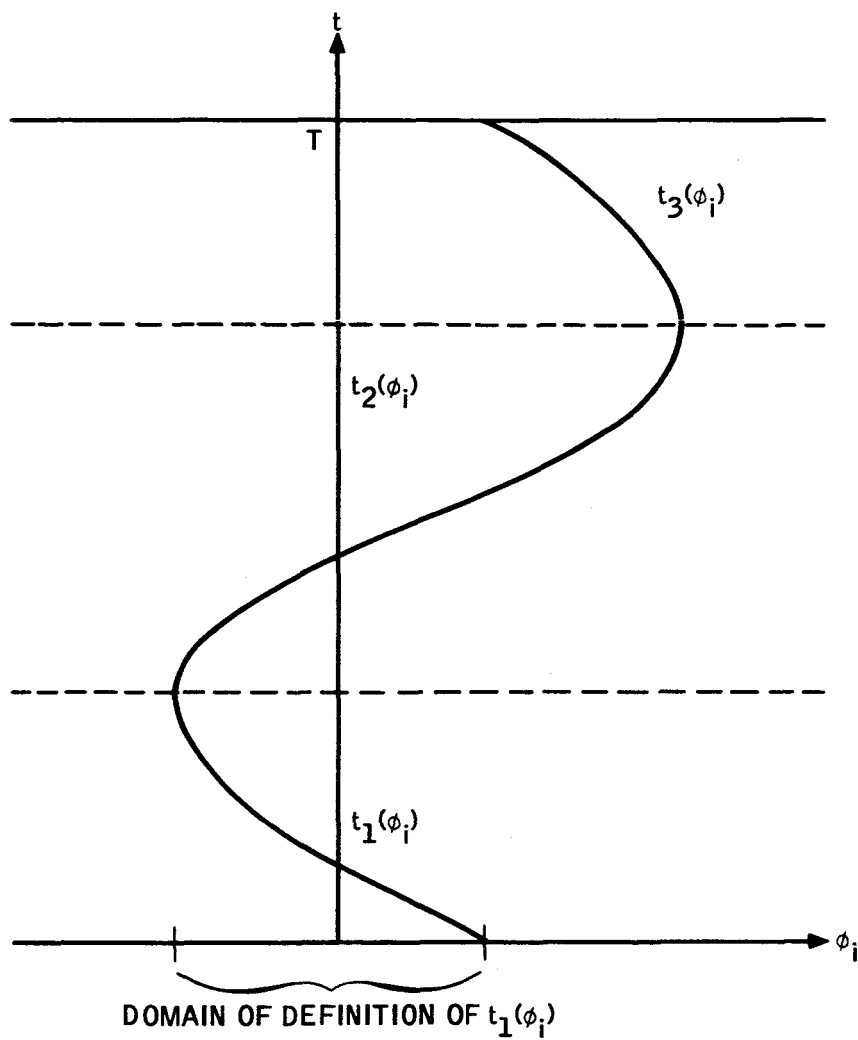Figure 3. The Function ϕ_i (t), Indicating Intervals of Monotonicity
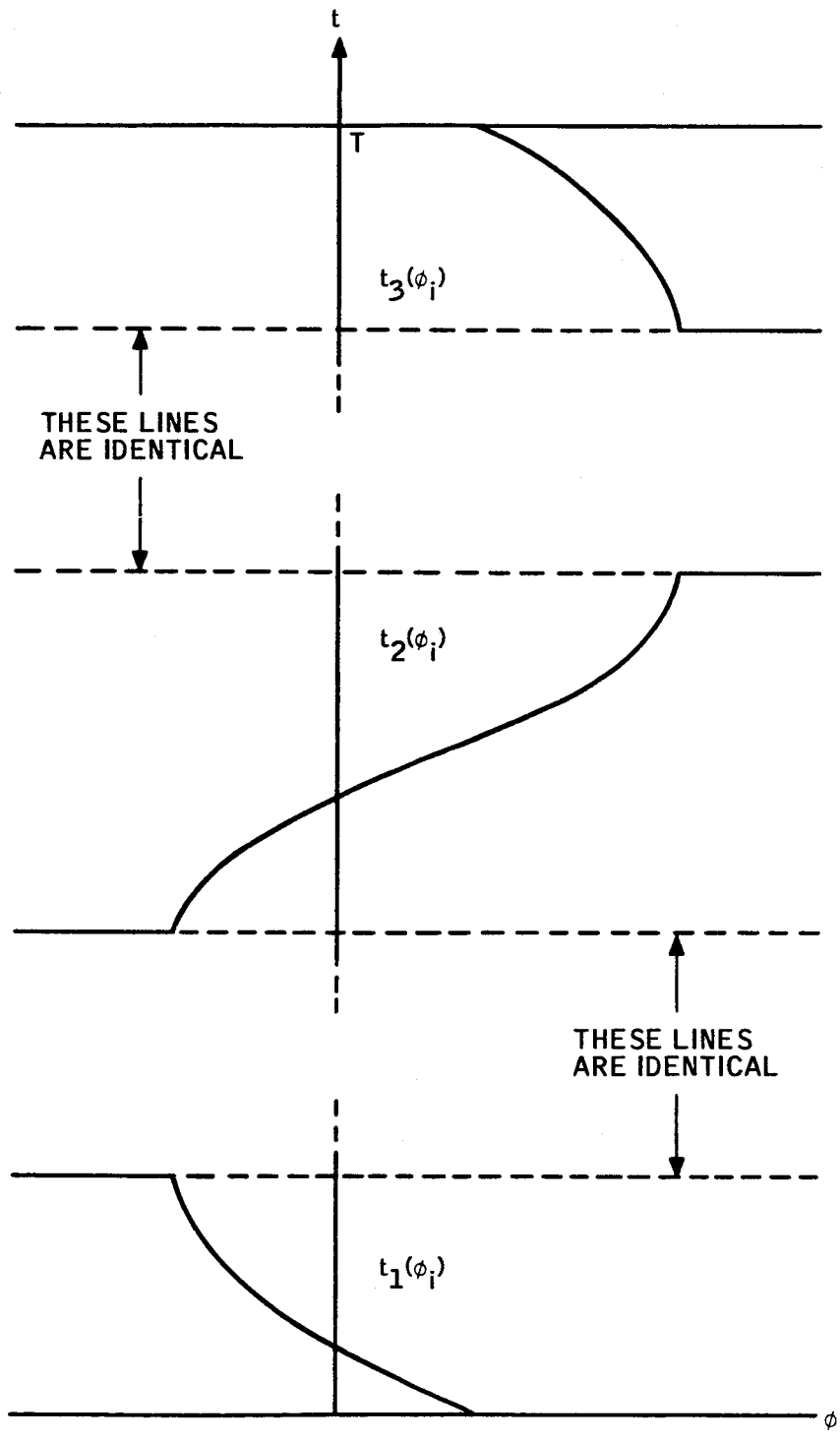
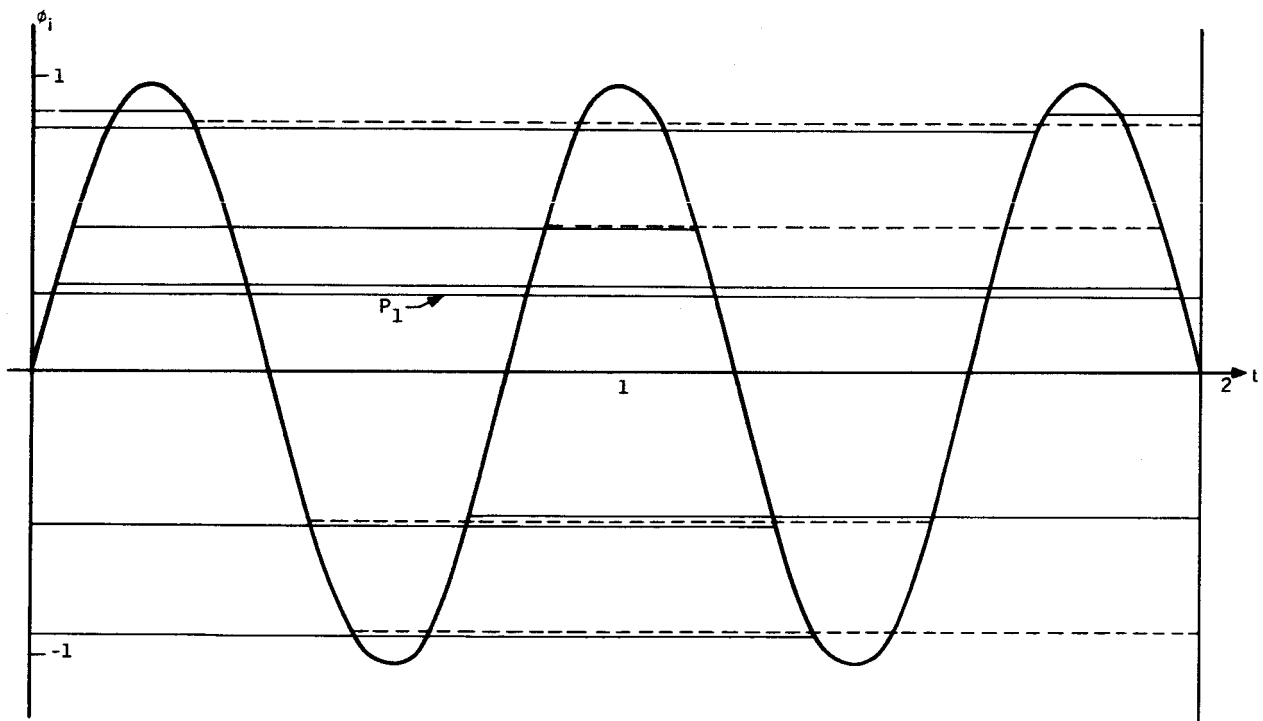Figure 4. The Inverse Functions $t_\sigma(\phi_i)$ of ϕ_i

Figure 5. Final Form of the Functions $t_\sigma(\phi_i)$

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NOTE: BROKEN LINES ARE ACTUALLY AT THE SAME LEVEL AS ADJACENT SOLID LINES

Figure 6. All Possible P_3 Intervals for the Function $\phi(t) = \sin\left(\frac{5\pi}{2}t\right)$ On the Interval $[0, 2]$
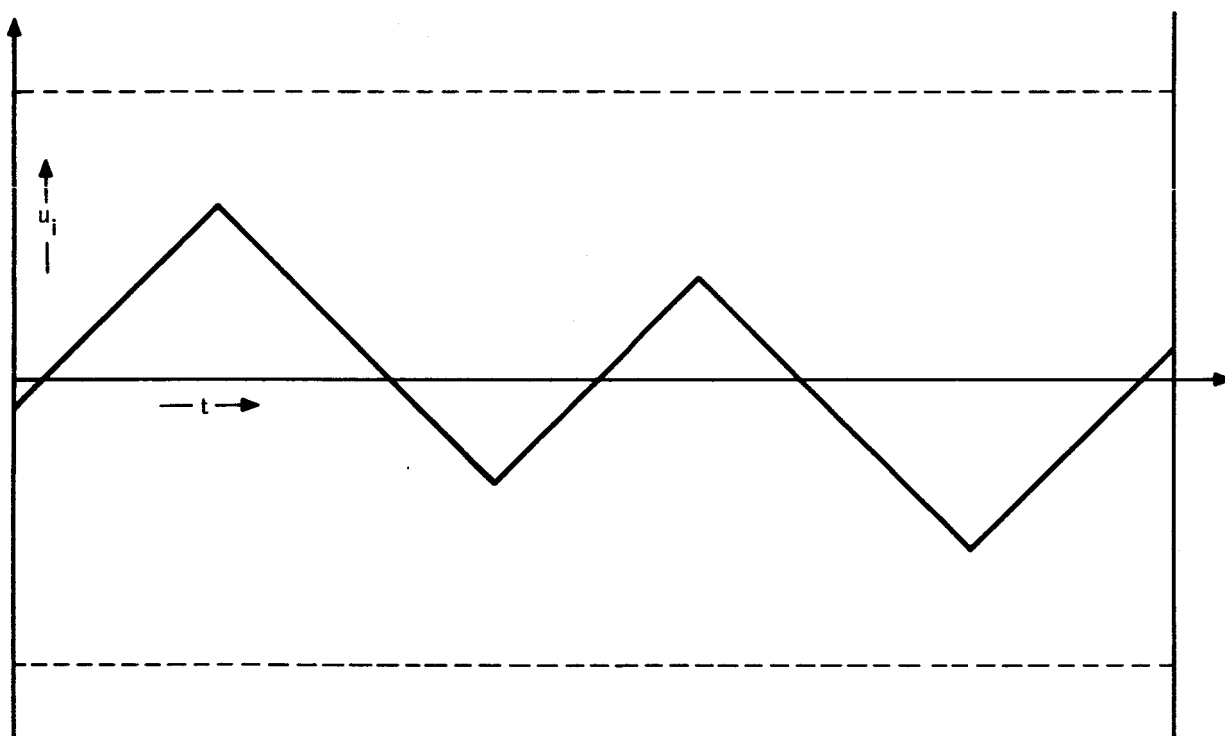
Figure 7. Extremal Control Constructed Using Results Shown on Figure 6